FLAT MITTAG-LEFFLER MODULES OVER COUNTABLE RINGS

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Abstract. We show that over any ring, the double Ext-orthogonal class to all flat Mittag-Leffler modules contains all countable direct limits of flat Mittag-Leffler modules. If the ring is countable, then the double orthogonal class consists precisely of all flat modules and we deduce, using a recent result of Šaroch and Trlifaj, that the class of flat Mittag-Leffler modules is not precovering in Mod-R unless R is right perfect.

Introduction

The notion of a Mittag-Leffler module was introduced by Raynaud and Gruson [10], who used the concept to prove a conjecture due to Grothendieck that the projectivity of infinitely generated modules over commutative rings is a local property. This is a crucial step for defining and working with infinitely generated vector bundles, as considered by Drinfeld in [3], where we also refer for more explanation.

The main step behind this geometrically motivated result is a completely general characterization of projective modules over any (in general non-commutative) ring R. Namely, one can show (consult [3]) that an R-module is projective if and only if M satisfies the following three conditions:

(1) M is flat,
(2) M is Mittag-Leffler,
(3) M is a direct sum of countably generated modules.

As mentioned by Drinfeld, the proof of projectivity of a given module might be non-constructive even in very simple cases, because it requires the Axiom of Choice. This applies for instance to the ring $\mathbb{Q}$ of rational numbers and the $\mathbb{Q}$-module $\mathbb{R}$ of real numbers. The main trouble there is condition (3). Thus, one might consider replacing projective modules by flat Mittag-Leffler modules (these are called "projective modules with a human face" in a preliminary version of [3]).

However, a surprising result in [5, §5] indicates that if one is interested in homological algebra, this might not be a good idea at all. Namely, the class of flat Mittag-Leffler abelian groups does not provide for precovers (sometimes also called right approximations). In the present paper, we

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use recent results due to Šaroch and Trlifaj [11] to show this is a much more general phenomenon and applies to many geometrically interesting examples. Namely, we prove in Theorem 6 that the class of flat Mittag-Leffler R-modules over a countable ring R is precovering if and only if R is a right perfect ring. Note that in that case the classes of projective modules, flat Mittag-Leffler modules and flat modules coincide, so the flat Mittag-Leffler precovers are just the projective ones.

1. Preliminaries

In this paper, R will always be an associative, not necessarily commutative ring with a unit. If not specified otherwise, a module will stand for a right R-module. We will denote by $\mathcal{D}$ the class of all modules which are flat and satisfy the Mittag-Leffler condition in the sense of [10, 8]:

Definition 1. M is called a *Mittag-Leffler* module if the canonical morphism $\rho : M \otimes_R \prod_{i\in I} Q_i \longrightarrow \prod_{i\in I} M \otimes_R Q_i$ is injective for each family $(Q_i \mid i \in I)$ of left R-modules.

A crucial closure property of the class $\mathcal{D}$ has been obtained in [8]:

Proposition 2. [8, Proposition 2.2] *Let R be a ring and $(F_i, u_{ji} : F_i \to F_j)$ be a direct system of modules from $\mathcal{D}$ indexed by $(I, \leq)$. Assume that for each increasing chain $(i_n \mid n < \omega)$ in I, the module $\varinjlim_{n<\omega} F_{i_n}$ belongs to $\mathcal{D}$. Then $M = \varinjlim_{i\in I} F_i$ belongs to $\mathcal{D}$.*

Let us look more closely at countable chains of modules and their limits. Recall that given a sequence of morphisms

$$F_0 \xrightarrow{u_0} F_1 \xrightarrow{u_1} F_2 \xrightarrow{u_2} F_3 \longrightarrow \dots$$

we have a short exact sequence

$$\eta : \qquad 0 \longrightarrow \bigoplus_{n<\omega} F_n \xrightarrow{\varphi} \bigoplus_{n<\omega} F_n \longrightarrow \varinjlim F_n \longrightarrow 0, \qquad (*)$$

such that φ is defined by $\varphi\iota_n = \iota_n - \iota_{n+1}u_n$, where $\iota_n : F_n \to \bigoplus F_m$ are the canonical inclusions. Note the following simple fact:

Lemma 3. *Given a chain $F_0 \overset{u_0}{\to} F_1 \overset{u_1}{\to} F_2 \to \dots$ of morphisms as above and a number $n_0 < \omega$, the middle term of $(*)$ decomposes as*

$$\bigoplus_{n<\omega} F_n = \varphi\left(\bigoplus_{m<n_0} F_m\right) \oplus \left(\bigoplus_{m\geq n_0} F_m\right).$$

Proof. Note that the module $\varphi(\bigoplus_{m<n_0} F_m)$ is generated by elements of the form

$$(0, \dots, 0, x_m, -u_m(x_m), 0 \dots) \in \bigoplus_{n<\omega} F_n,$$

where $m < n_0$ and $x_m \in F_m$. It follows easily that one can uniquely express each $y = (y_n) \in \bigoplus_{n<\omega} F_n$ as $y = z + w$, where $z \in \varphi\left(\bigoplus_{m<n_0} F_m\right)$ and $w \in \bigoplus_{m\geq n_0} F_m$. Namely, we take $z = (y_0, \dots, y_{n_0-1}, y'_{n_0}, 0, 0, \dots)$ with $-y'_{n_0} = u_{n_0-1}(y_{n_0-1}) + u_{n_0-1}u_{n_0-2}(y_{n_0-2}) + \dots + u_{n_0-1}u_{n_0-2}\dots u_1u_0(y_0)$ and $w = y - z \in \bigoplus_{m\geq n_0} F_m$. $\square$

We will also need a few simple results concerning infinite combinatorics, starting with a well known lemma.

Lemma 4. *For any cardinal μ there is a cardinal $\lambda \geq \mu$ such that $\lambda^{\aleph_0} = 2^\lambda$.*

Proof. We refer for instance to [7, Lemma 3.1]. For the reader's convenience, we recall how to construct λ. We put $\mu_0 = \mu$ and for each $n < \omega$ inductively construct $\mu_{n+1} = 2^{\mu_n}$. Then $\lambda = \sup_{n<\omega} \mu_n$ has the required property, see for example [9, p. 50, fact (6.21)]. □

The next lemma deals with a construction of a large family of "almost disjoint" maps $f : \omega \to \lambda$. The result is well known in the literature and it has many different proofs. We refer for instance to [2, Lemma 2.3] or [4, Proposition II.5.5].

Lemma 5. *Let λ be an infinite cardinal. Then there is a subset $J \subseteq \lambda^\omega$ of cardinality $\lambda^{\aleph_0}$ such that for any pair of distinct maps $f, g : \omega \to \lambda$ of J, the set formed by the $x \in \omega$ on which the values $f(x)$ and $g(x)$ coincide is a finite initial segment of ω.*

Proof. Consider the tree T of the finite sequences of elements of λ, i.e. $T = \{t : n \to \lambda \mid n < \omega\}$. Since λ is infinite, we have $|T| = \left|\bigcup_{n<\omega} \lambda^n\right| = \lambda$, so there is a bijection $b : T \to \lambda$. For every map $f : \omega \to \lambda$ denote by $A_f : \omega \to T$ the induced map which sends $n < \omega$ to $f \restriction n \in T$. Clearly, if f, g are two different maps in λ^ω, the values of A_f and A_g coincide only on a finite initial segment of ω. Now, we can put $J = \{b \circ A_f \mid f \in \lambda^\omega\}$. □

2. Main result

Now we are in a position to state our main result, which is inspired by [8, §5]. It will be proved by using a cardinal argument similar to the one in [2, Proposition 2.5]. Note that the result sharpens [11, Theorem 2.9] by removing the additional set-theoretical assumption of Singular Cardinal Hypothesis, and also [8, Corollaries 7.6 and 7.7] by removing the assumption that $\mathcal{D}$ is closed under products.

Regarding the notation and terminology, given a class $\mathcal{C} \subseteq \text{Mod-}R$, we put $\mathcal{C}^\perp = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(\mathcal{C}, M) = 0\}$ and dually ${}^\perp\mathcal{C} = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, \mathcal{C}) = 0\}$. Recall that a module is called *cotorsion* if it cannot be non-trivially extended by a flat module.

We recall also the notion of a precover, or sometimes called right approximation. If $\mathcal{X}$ is any class of modules and $M \in \text{Mod-}R$, a homomorphism $f\colon X \to M$ is called an *$\mathcal{X}$-precover* of M if $X \in \mathcal{X}$ and for every homomorphism $f' \in \text{Hom}_R(X', M)$ with $X' \in \mathcal{X}$ there exists a homomorphism $g\colon X' \to X$ such that $f' = fg$. The class $\mathcal{X}$ is called *precovering* if each $M \in \text{Mod-}R$ admits an $\mathcal{X}$-precover.

Theorem 6. *Let R be a ring and $\mathcal{D}$ be the class of all flat Mittag-Leffler right R-modules. Given any countable chain*

$$F_0 \xrightarrow{u_0} F_1 \xrightarrow{u_1} F_2 \xrightarrow{u_2} F_3 \longrightarrow \dots$$

of morphisms such that $F_n \in \mathcal{D}$ for all $n < \omega$, we have $\varinjlim F_n \in {}^\perp(\mathcal{D}^\perp)$. If, moreover, R is a countable ring, then the following hold:

(1) *$\mathcal{D}^\perp$ is precisely the class of all cotorsion modules.*
(2) *$\mathcal{D}$ is a precovering class in* Mod-R *if and only if R is right perfect.*

Proof. Assume we have a countable direct system (F_n, u_n) as above, put $F = \varinjlim F_n$, and fix a module $C \in \mathcal{D}^\perp$. We must prove that $\operatorname{Ext}^1_R(F, C) = 0$.

Let us fix an infinite cardinal λ, depending on C, such that we have $\lambda \geq |\operatorname{Hom}_R(F_n, C)|$ for each $n < \omega$ and $\lambda^{\aleph_0} = 2^\lambda$; we can do this using Lemma 4. Applying Lemma 5, we find a subset $J \subseteq \lambda^\omega$ of cardinality 2^λ such that the values of each pair $f, g : \omega \to \lambda$ of distinct elements of J coincide only on a finite initial segment of ω. We claim that there is a short exact sequence of the form

$$0 \longrightarrow P \longrightarrow E \longrightarrow F^{(2^\lambda)} \longrightarrow 0 \tag{$\dagger$}$$

such that $E \in \mathcal{D}$ and $|\operatorname{Hom}_R(P, C)| \leq 2^\lambda$.

Let us construct such a sequence. First, denote for each $\alpha < \lambda$ by $F_{n,\alpha}$ a copy of F_n, and by P the direct sum $\bigoplus F_{n,\alpha}$ taken over all pairs (n, α) such that $n < \omega$ and $\alpha = f(n)$ for some $f \in J$. Note that P is a summand in $\bigoplus_{n<\omega} F_n^{(\lambda)}$, so we have

$$|\operatorname{Hom}_R(P, C)| \leq \Big|\operatorname{Hom}_R\big(\bigoplus F_n^{(\lambda)}, C\big)\Big| \leq \prod_{n<\omega} |\operatorname{Hom}_R(F_n, C)|^\lambda \leq \lambda^{\omega\times\lambda} = 2^\lambda.$$

Next, we will construct E. Given $f \in J$, let

$$\iota_f : \bigoplus_{n<\omega} F_n \longrightarrow P$$

be the split inclusion which sends each F_n to $F_{n,f(n)}$. Using the short exact sequence $(*)$ from page 2, we can extend P by F via the following pushout diagram:

$$\begin{array}{llcccccccc}
\eta: & 0 & \longrightarrow & \bigoplus_{n<\omega} F_n & \xrightarrow{\varphi} & \bigoplus_{n<\omega} F_n & \longrightarrow & F & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle \iota_f} & & \Big\downarrow{\scriptstyle \vartheta_f} & & \Big\| & & \\
\varepsilon_f: & 0 & \longrightarrow & P & \xrightarrow{\subseteq} & E_f & \longrightarrow & F & \longrightarrow & 0
\end{array} \tag{Δ}$$

Now, we can put these extensions for all $f \in J$ together. Namely, let $\sigma : P^{(J)} \to P$ be the summing map and consider the pushout diagram:

$$\begin{array}{llcccccccc}
\bigoplus \varepsilon_f: & 0 & \longrightarrow & P^{(J)} & \longrightarrow & \bigoplus_{f\in J} E_f & \xrightarrow{\rho} & F^{(J)} & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \pi} & & \Big\| & & \\
\varepsilon: & 0 & \longrightarrow & P & \longrightarrow & E & \longrightarrow & F^{(J)} & \longrightarrow & 0
\end{array}$$

For each $g \in J$, the composition of the canonical inclusion $\nu_g \colon E_g \to \bigoplus_{f\in J} E_f$ with the morphism π yields a monomorphism $E_g \to E$. In fact, if $y \in E_g$ is such that $\pi\nu_g(y) = 0$, then $\rho(\nu_g(y)) = 0$, hence the exact sequence ε_g gives that y is in the image of P and the composition of the canonical embedding $\mu_g \colon P \to P^{(J)}$ with the morphism σ is a monomorphism. From now on we shall without loss of generality view these monomorphisms $E_g \to E$ as inclusions.

To prove the existence of (†), it suffices to show that $E \in \mathcal{D}$ in ε. To this end, denote for any subset $S \subseteq J$ by M_S the module

$$M_S = \sum_{f \in S} \operatorname{Im} \vartheta_f \quad (\subseteq E, \text{ see diagram } (\Delta))$$

Then the family $(M_S \mid S \subseteq J \,\&\, |S| \leq \aleph_0)$ with obvious inclusions forms a direct system and we claim that its union is the whole of E. Indeed, it is straightforward to check, using diagram (Δ) and the construction of the embeddings $E_g \subseteq E$, that $E = P + \sum_{f \in J} \operatorname{Im} \vartheta_f$. Further, the left hand square of diagram (Δ) is a pull-back, which implies $P \cap \operatorname{Im} \vartheta_f = \operatorname{Im} \iota_f$ and

$$P \cap \sum_{f \in J} \operatorname{Im} \vartheta_f \supseteq \sum_{f \in J} \operatorname{Im} \iota_f = P.$$

Thus, $E = \sum_{f \in J} \operatorname{Im} \vartheta_f$ and the claim is proved.

Moreover, the union of any chain $M_{S_0} \subseteq M_{S_1} \subseteq M_{S_2} \subseteq \dots$ from the direct system belongs to the direct system again. Therefore, if we prove that $M_S \in \mathcal{D}$ for each countable $S \subseteq J$, it will follow from Proposition 2 that $E \in \mathcal{D}$. Our task is then reduced to prove the following lemma:

Lemma 7. *With the notation as above, the following hold:*

(1) *Given $S \subseteq T \subseteq J$ with S and T finite and such that $|T| = |S|+1$, the inclusion $M_S \subseteq M_T$ splits and there is $n_0 < \omega$ such that $M_T/M_S \cong \bigoplus_{m \geq n_0} F_m$.*
(2) *Given a countable subset $S \subseteq J$, the module M_S is isomorphic to a countable direct sum with each summand isomorphic to some F_n, $n < \omega$. In particular, $M_S \in \mathcal{D}$.*

Proof. Let us focus on (1) since (2) is an immediate consequence. Denote by $f : \omega \to \lambda$ the single element of $T \setminus S$, and let $n_0 < \omega$ be the smallest number such that $f(n_0) \neq g(n_0)$ for each $g \in S$.

We claim that the following are satisfied by the construction:

$$M_S \cap \operatorname{Im} \vartheta_f = \iota_f \left(\bigoplus_{m < n_0} F_m \right) = \vartheta_f \circ \varphi \left(\bigoplus_{m < n_0} F_m \right).$$

The second equality holds simply because $\vartheta_f \circ \varphi = \iota_f$ by diagram (Δ). For the first, note that $M_S \cap \operatorname{Im} \vartheta_f$ as a submodule of E, is contained in P. Since $P \cap \operatorname{Im} \vartheta_f = \operatorname{Im} \iota_f$, we have

$$M_S \cap \operatorname{Im} \vartheta_f = \left(\sum_{g \in S} \operatorname{Im} \vartheta_g \right) \cap \operatorname{Im} \iota_f = \left(\sum_{g \in S} \operatorname{Im} \iota_g \right) \cap \operatorname{Im} \iota_f = \iota_f \left(\bigoplus_{m < n_0} F_m \right),$$

by the construction of P. This proves the claim.

Invoking Lemma 3, we further deduce that

$$\operatorname{Im} \vartheta_f = \iota_f \left(\bigoplus_{m < n_0} F_m \right) \oplus \vartheta_f \left(\bigoplus_{m \geq n_0} F_m \right).$$

In particular, the inclusion $M_S \cap \operatorname{Im}\vartheta_f \subseteq \operatorname{Im}\vartheta_f$ splits and so does the inclusion $M_S \subseteq M_S + \operatorname{Im}\vartheta_f = M_T$. Moreover, we have the isomorphisms

$$M_T/M_S = (M_S + \operatorname{Im}\vartheta_f)/M_S \cong \operatorname{Im}\vartheta_f/(M_S \cap \operatorname{Im}\vartheta_f) \cong \bigoplus_{m \geq n_0} F_m,$$

which finishes the proof of the lemma. □

Having established the existence of (†) such that $E \in \mathcal{D}$ and $|\operatorname{Hom}_R(P, C)| \leq 2^\lambda$, let us apply $\operatorname{Hom}_R(-, C)$ on (†). Since $C \in \mathcal{D}^\perp$ by assumption, we get an exact sequence

$$\operatorname{Hom}_R(P, C) \longrightarrow \operatorname{Ext}^1_R(F^{(2^\lambda)}, C) \longrightarrow 0.$$

Suppose now that $\operatorname{Ext}^1_R(F, C) \neq 0$. Then we would have $|\operatorname{Ext}^1_R(F^{(2^\lambda)}, C)| \geq 2^{2^\lambda}$, which would contradict the fact that $|\operatorname{Hom}_R(P, C)| \leq 2^\lambda$. Hence $\operatorname{Ext}^1_R(F, C) = 0$ as desired.

To finish the proof of Theorem 6, suppose R is a countable ring. Since each $F \in \mathcal{D}$ is flat, $\mathcal{D}^\perp$ contains all cotorsion modules. On the other hand, if C is not cotorsion, there is a countable flat module F such that $\operatorname{Ext}^1_R(F, C) \neq 0$; see for instance [6, Theorems 4.1.1 and 3.2.9]. By the first part of Theorem 6, we know that $F \in {}^\perp(\mathcal{D}^\perp)$, so $C \notin \mathcal{D}^\perp$. Hence $\mathcal{D}^\perp$ consists precisely of cotorsion modules.

The fact that $\mathcal{D}$ is not precovering unless R is right perfect (and $\mathcal{D}$ is then the class of projective modules) follows directly from [11, Theorem 2.10]. This finishes the proof of Theorem 6. □

Remark 8. The proof of Theorem 6 is to some extent constructive. Namely, if R is a countable ring and C is a module which is not cotorsion, the theorem gives us a recipe how to construct $E \in \mathcal{D}$ such that $\operatorname{Ext}^1_R(E, C) \neq 0$, and it allows us to estimate the size of E based on the size of C. Note that if R is non-perfect, the size of E must grow with the size of C. This is because for any set $\mathcal{S} \subseteq \mathcal{D}$, we have ${}^\perp(\mathcal{S}^\perp) \subseteq \mathcal{D} \subsetneqq \text{Flat-}R$ by [1, Proposition 1.9] and [6, Corollary 3.2.3].

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